Nonlinear physics: Hamiltonian Chaos

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Outline

- **Hamiltonian systems (Example Hénon-Heiles system)**
  - ✓**Equations of motion**
  - ✓**Chaos**
  - ✓**Poincaré Surface of Section**
  - ✓**Variational equations**
  - ✓**Lyapunov exponents**

Autonomous Hamiltonian systems

Consider an N degree of freedom autonomous Hamiltonian system having a Hamiltonian function of the form:

$$H(\overbrace{q_1,q_2,\ldots,q_N}^{\text{positions}}, \overbrace{p_1,p_2,\ldots,p_N}^{\text{momenta}})$$

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The time evolution of an orbit (trajectory) with initial condition

$$P(0)=(q_1(0),\ q_2(0),\ldots,q_N(0),\ p_1(0),\ p_2(0),\ldots,p_N(0))$$

is governed by the Hamilton's equations of motion

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}\ ,\qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

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Phase space: the 2N dimensional space defined by variables $q_1,q_2,\ldots,q_N,\ p_1,p_2,\ldots,p_N$

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(Received 7 August 1963)

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Hamilton's equations of motion:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \Rightarrow \begin{cases} \dot{x} = p_x \\ \dot{y} = p_y \\ \dot{p}_x = -x - 2xy \\ \dot{p}_y = -y - x^2 + y^2 \end{cases}$$

Chaos

Definition [Devaney (1989)]

Let V be a set and $f : V \to V$ a map on this set.

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2. **f is topologically transitive.**
3. **periodic points are dense in V.**

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There exist points arbitrarily close to $\mathbf{x}$ which eventually separate from $\mathbf{x}$ by at least δ under iterations of $\mathbf{f}$.

Not all points near $\mathbf{x}$ need eventually move away from $\mathbf{x}$ under iteration, but there must be at least one such point in every neighborhood of $\mathbf{x}$.

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Consequently, the dynamical system cannot be decomposed into two disjoint invariant open sets.

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3. An element of regularity because it has periodic points which are dense.

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2. **Indecomposability** because it cannot be decomposed into noninteracting subsystems due to topological transitivity
3. **An element of regularity** because it has periodic points which are dense.

Usually, in physics and applied sciences, people use the notion of chaos in relation to the sensitive dependence on initial conditions.

Regular vs Chaotic orbits

Hénon-Heiles system

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + x^2 y - \frac{1}{3} y^3$$

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Orbit **Perturbed**

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		Orbit	**Perturbed**
t=	100	x= 0.13299571833307644	0.13299571833726306

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			Orbit	Perturbed
t=	100	x=	0.13299571833307644	0.13299571833727063064
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t=	5000	x= 0.37699928389102310	0.37699928370156576
t=	10000	x=-0.15909458335685522 4	-0.15909458334126030 9
t=	50000	x= 0.10199240073995576 0	0.10199240025396132 1

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t=	10000	x=-0.15909458335685524	-0.15909458334126030 9
t=	50000	x= 0.10199240073995576 0	0.10199240025396132 1
t=	100000	x=-0.38112053374651178 0	-0.38112053332725887 0

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t=	50000	x= 0.10199240073995576	0.10199240025396132
t=	100000	x=-0.38112053374651178	-0.38112053332725887
t=	100	x= 0.09027281773516783	0.09027282135576866

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t=	100	x= 0.13299571833307644	0.13299571833726306 4
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t=	50000	x= 0.10199240073995576 0	0.10199240025396132 1
t=	100000	x=-0.38112053374651178 0	-0.38112053332725887 0
t=	100	x= 0.09027281773516783 5	0.09027282135576866 8
t=	200	x= 0.29503168748224928 3	0.29503188485862563 7

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t=100000	x=-0.38112053374651178	-0.38112053332725887
t= 100	x= 0.09027281773516783	0.09027282135576866
t= 200	x= 0.29503168748224928	0.29503188485862563
t= 300	x= 0.51522633010945018	0.51522544048069329

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	Orbit	Perturbed
t= 100	x= 0.13299571833307644	0.13299571833763064
t= 5000	x= 0.37699928389102310	0.37699928370156576
t= 10000	x=-0.15909458335685224	-0.15909458334126030
t= 50000	x= 0.10199240073995576	0.10199240025396132
t=100000	x=-0.38112053374651178	-0.38112053332725887
t= 100	x= 0.09027281773516783	0.09027282135576866
t= 200	x= 0.29503168748224928	0.29503188485862563
t= 300	x= 0.51522633010945018	0.51522544048069329
t= 400	x= 0.06344188934742586	0.06135955855100834

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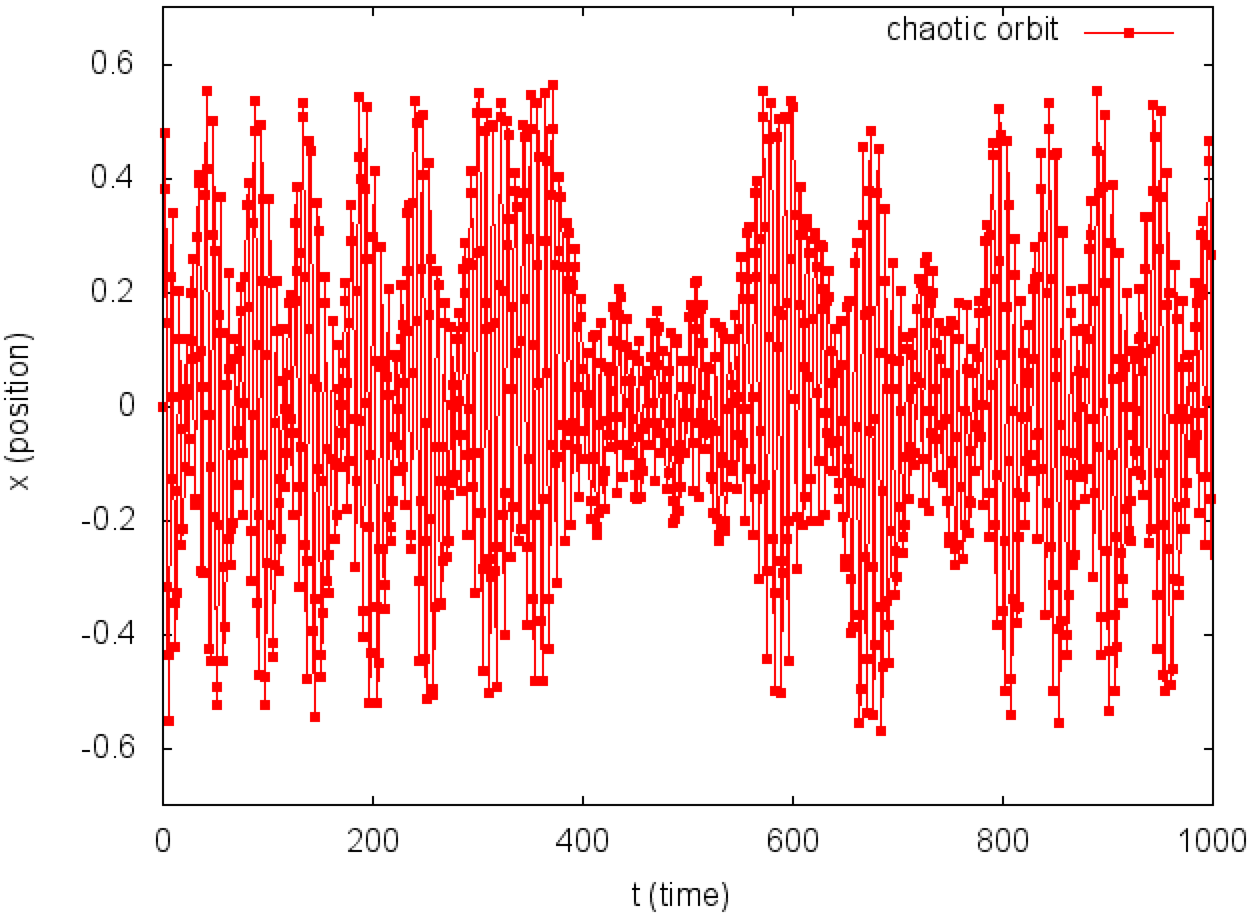
t=	50000	x= 0.101992400739955760	0.101992400253961321
t=	100000	x=-0.381120533746511780	-0.381120533327258870
t=	100	x= 0.090272817735167835	0.090272821355768668
t=	200	x= 0.295031687482249283	0.295031884858625637
t=	300	x= 0.515226330109450181	0.515225440480693297
t=	400	x= 0.063441889347425867	0.061359558551008345
t=	500	x= 0.078357719290523528	-0.270811022674341095

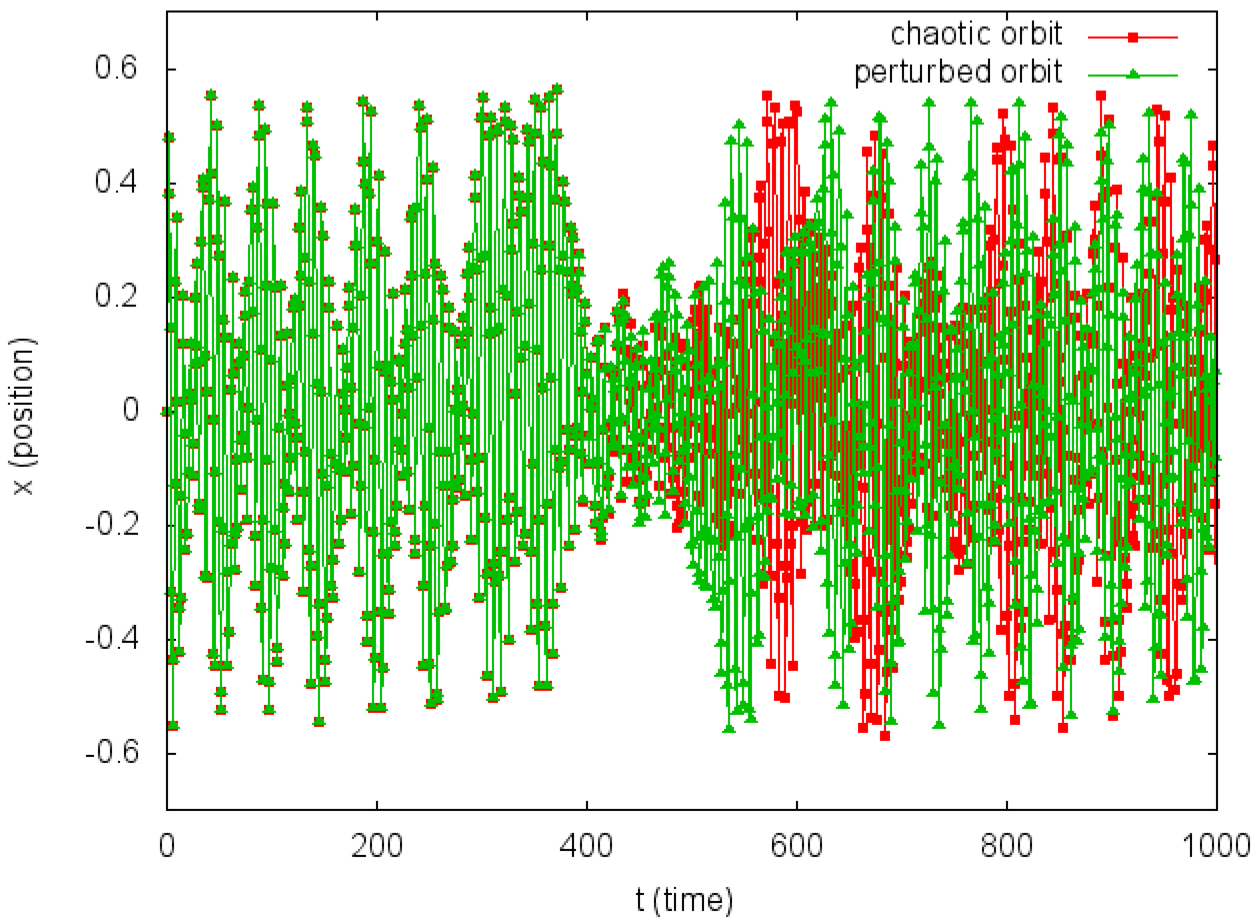
Regular vs Chaotic orbits

Chaotic orbit



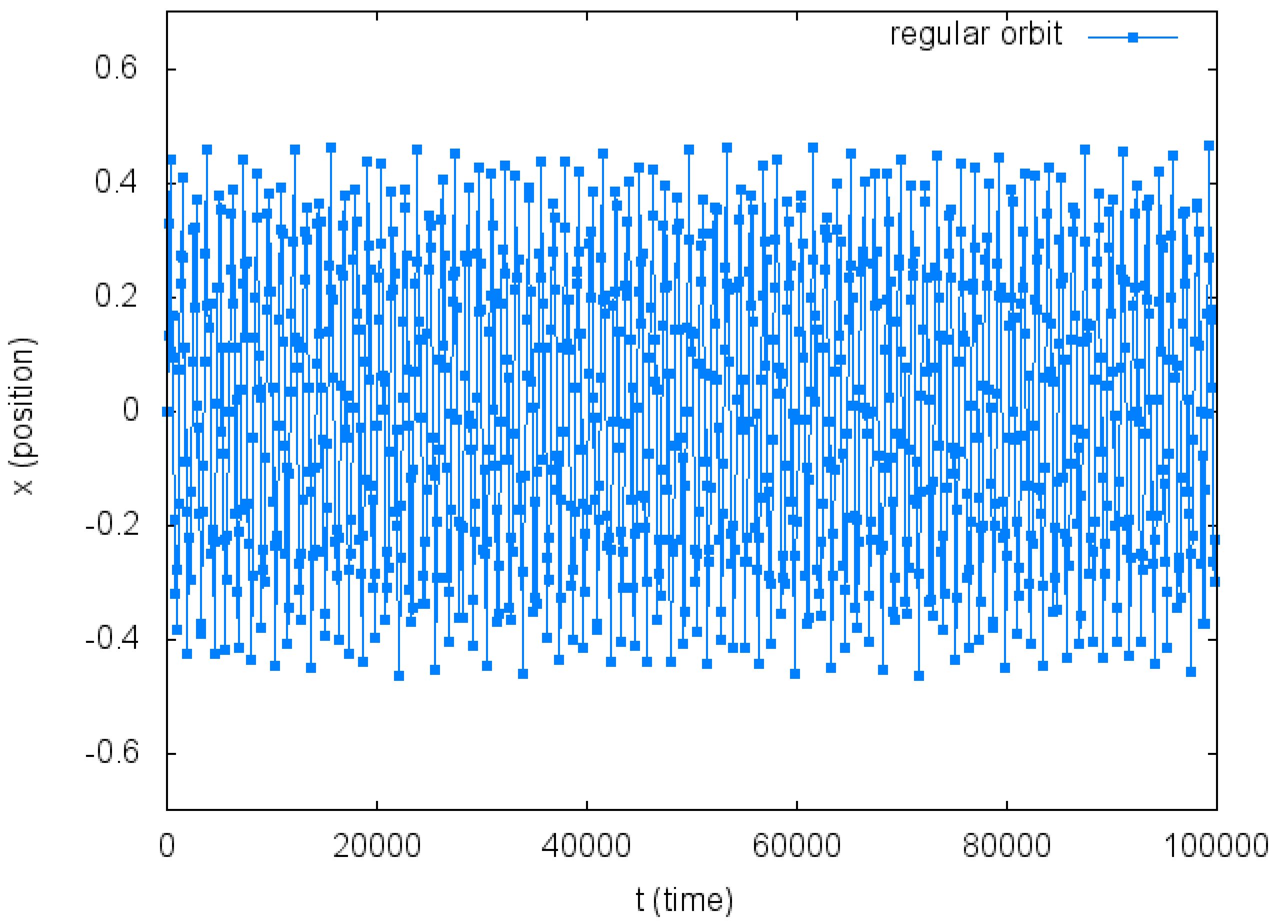
Regular vs Chaotic orbits

Chaotic orbit and its perturbation



Regular vs Chaotic orbits

Regular orbit



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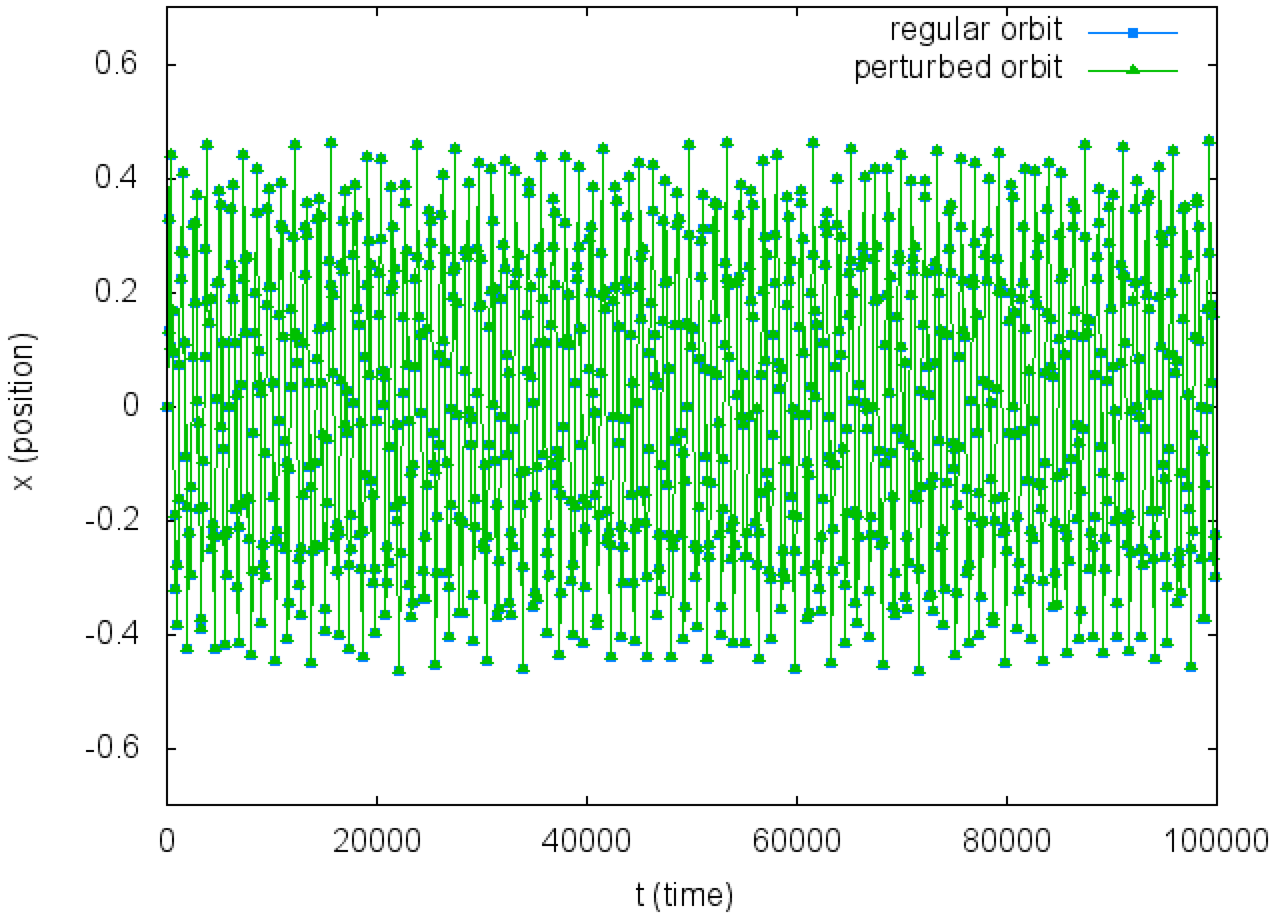
Regular orbit and its perturbation



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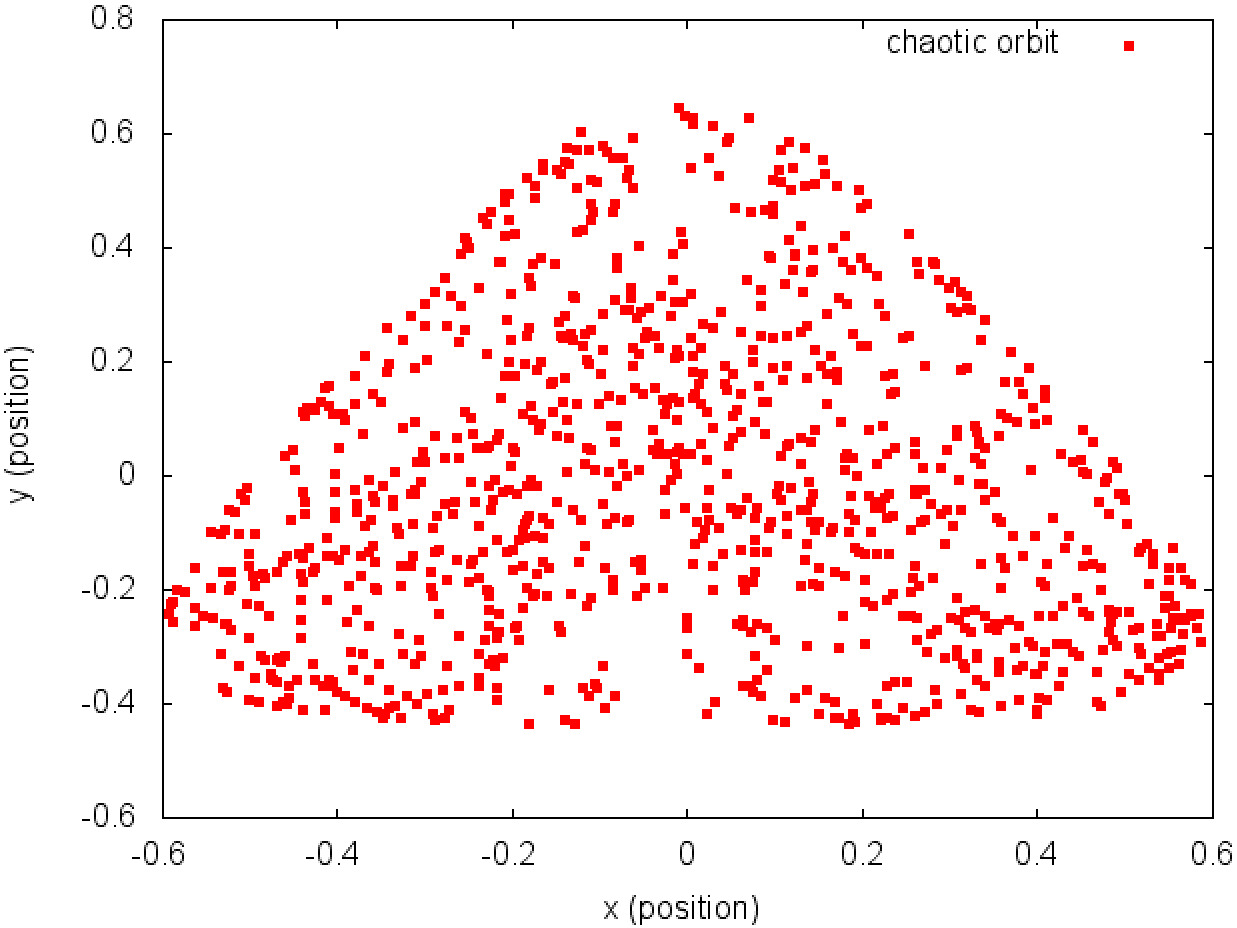
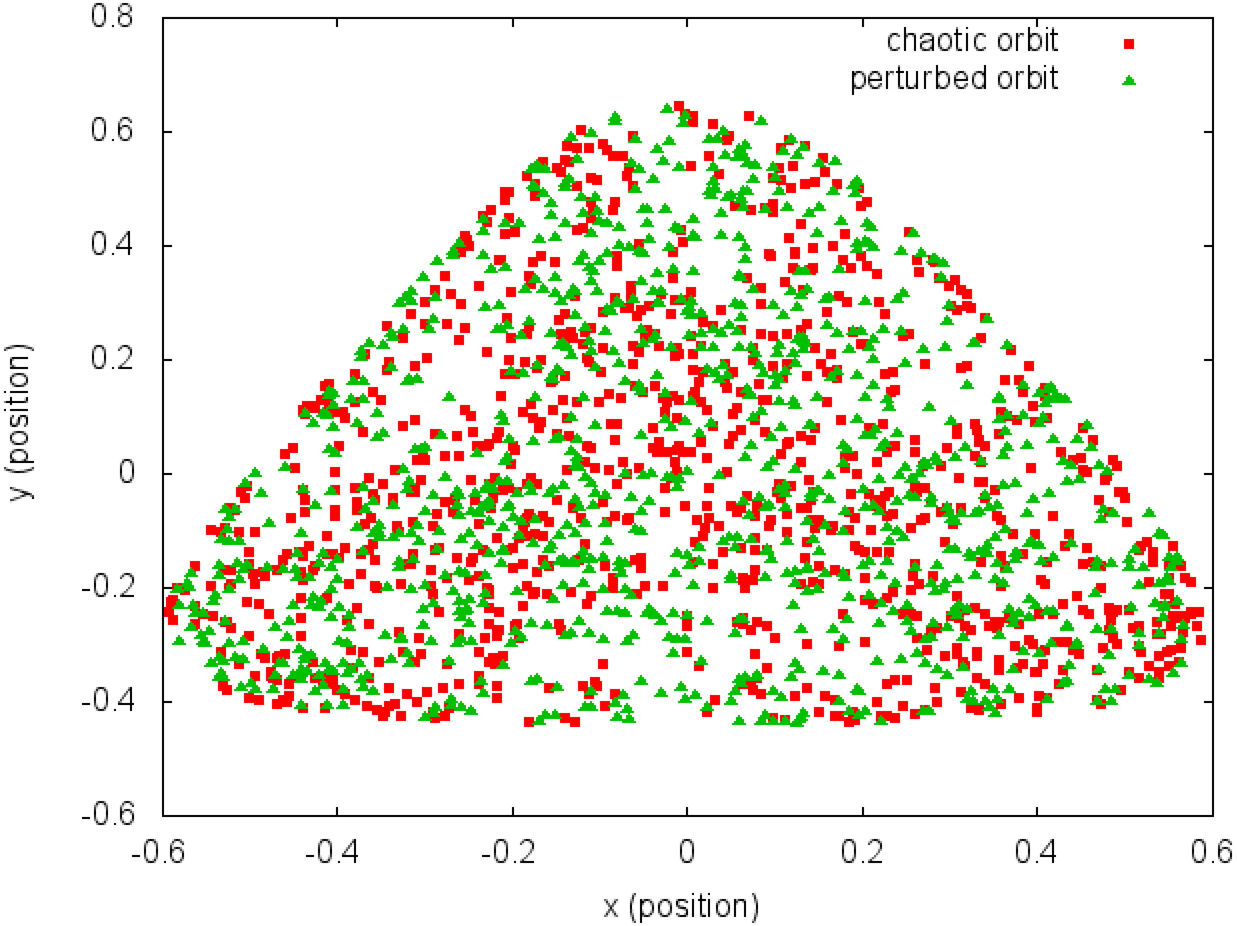


Results for $0 \leq t \leq 10^5$

Regular vs Chaotic orbits

Chaotic orbit and its perturbation



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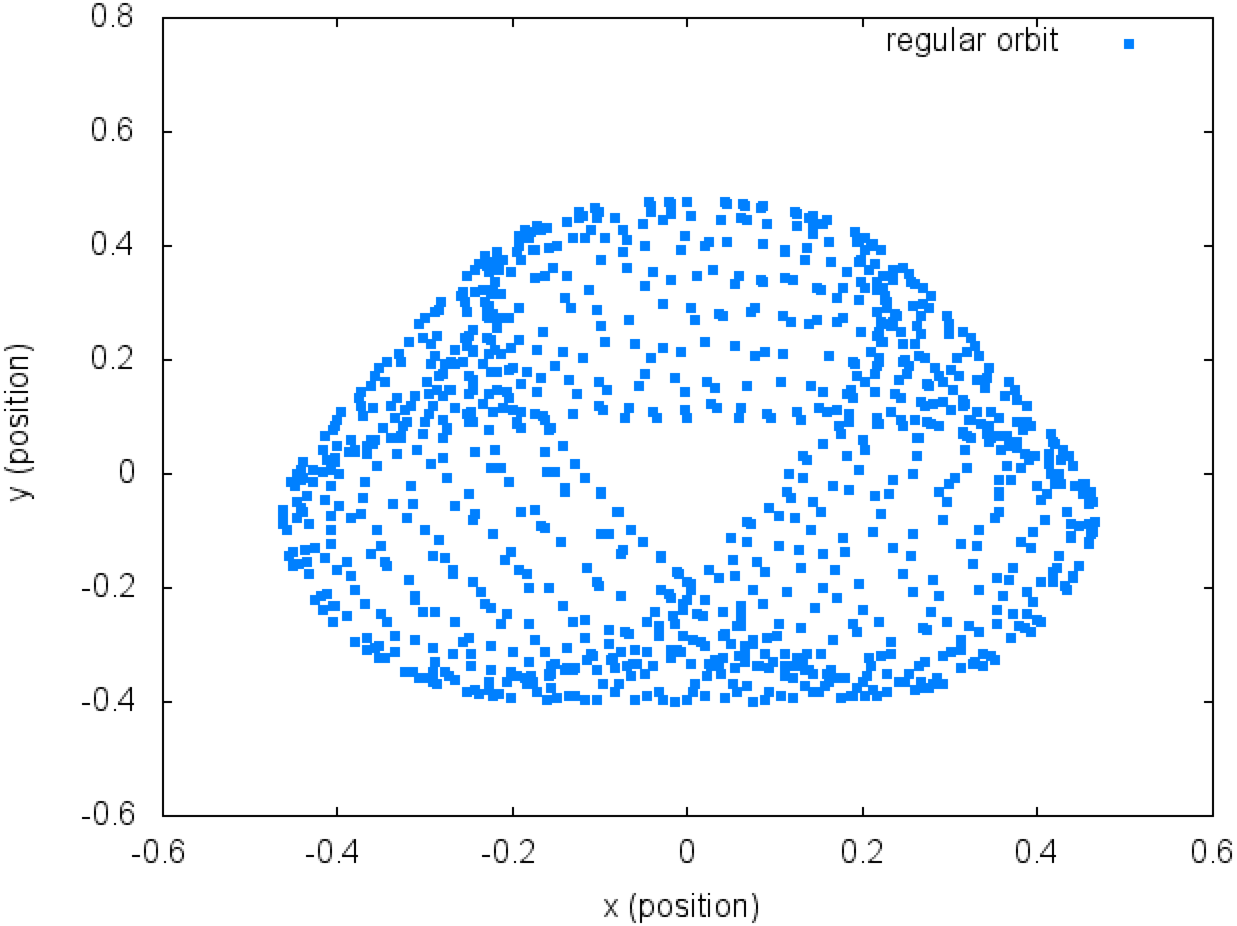

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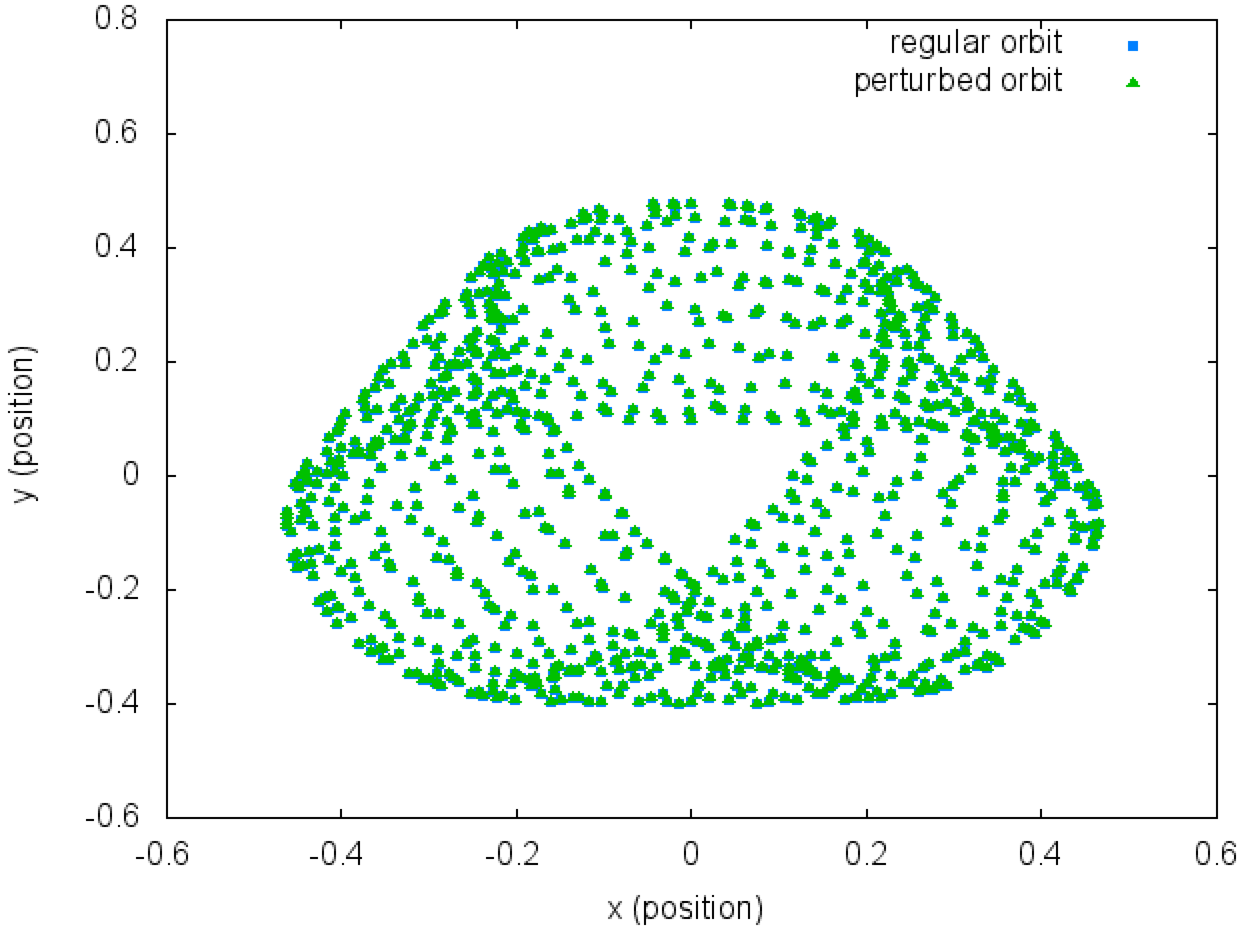

Results for $0 \leq t \leq 10^5$

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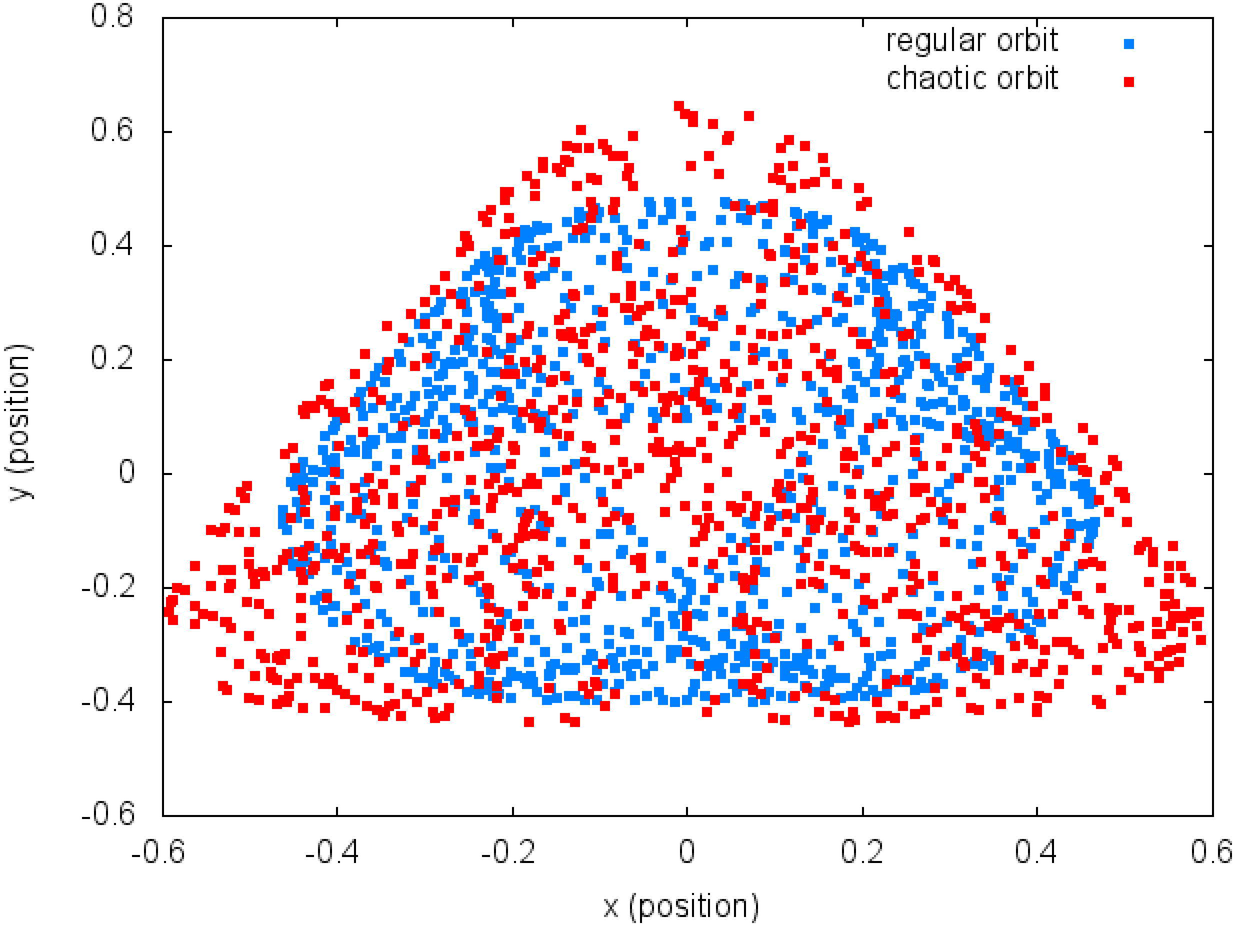
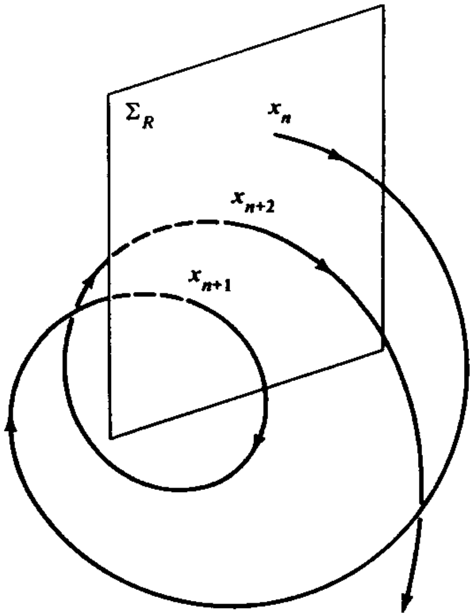
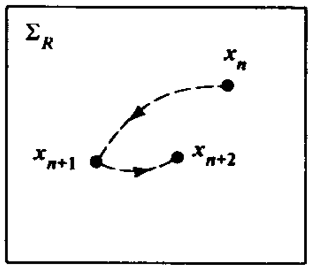


Results for $0 \leq t \leq 10^5$

Poincaré Surface of Section (PSS)

We can constrain the study of an N+1 degree of freedom Hamiltonian system to a 2N-dimensional subspace of the general phase space.





Lieberman & Lichtenberg, 1992, *Regular and Chaotic Dynamics*, Springer.

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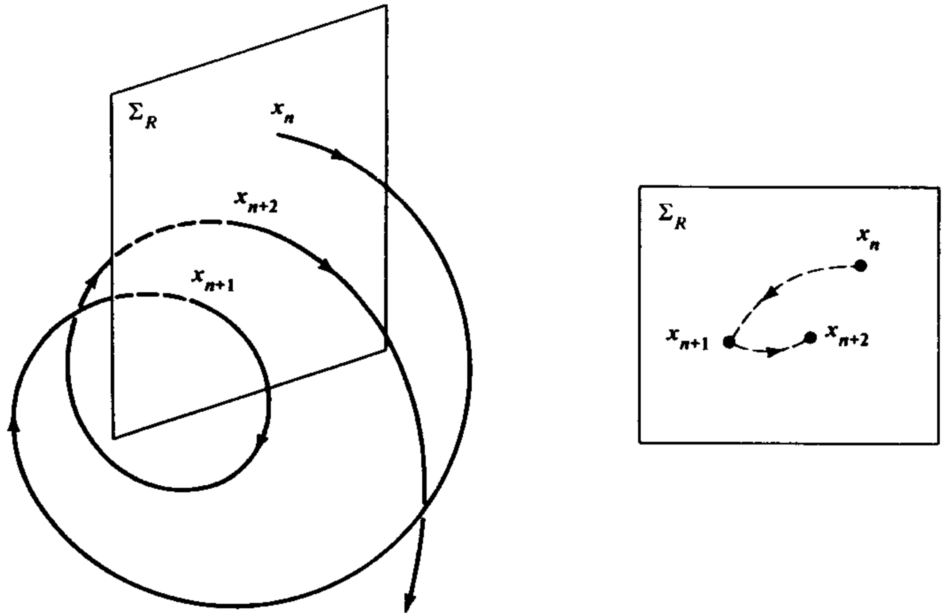

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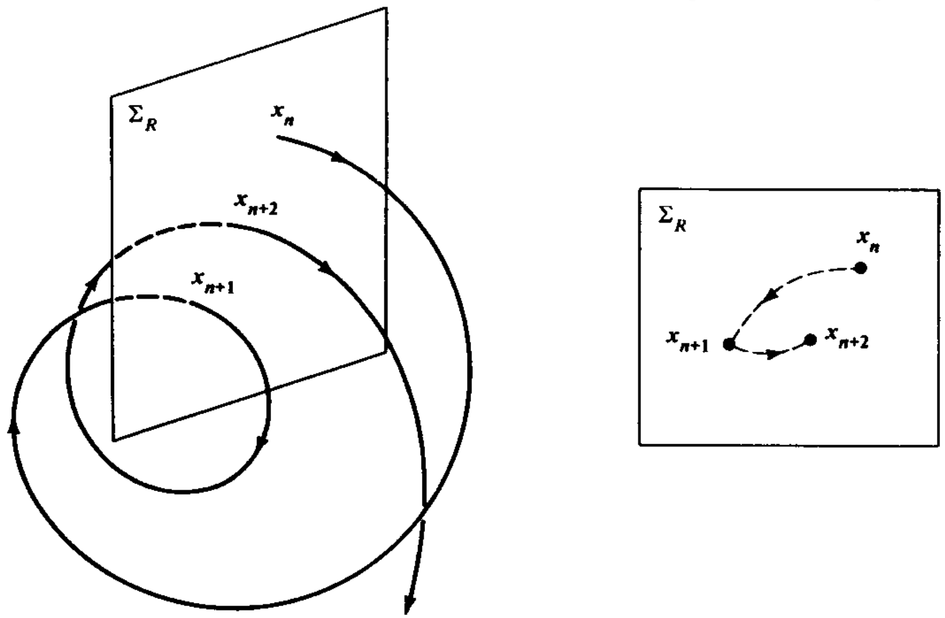

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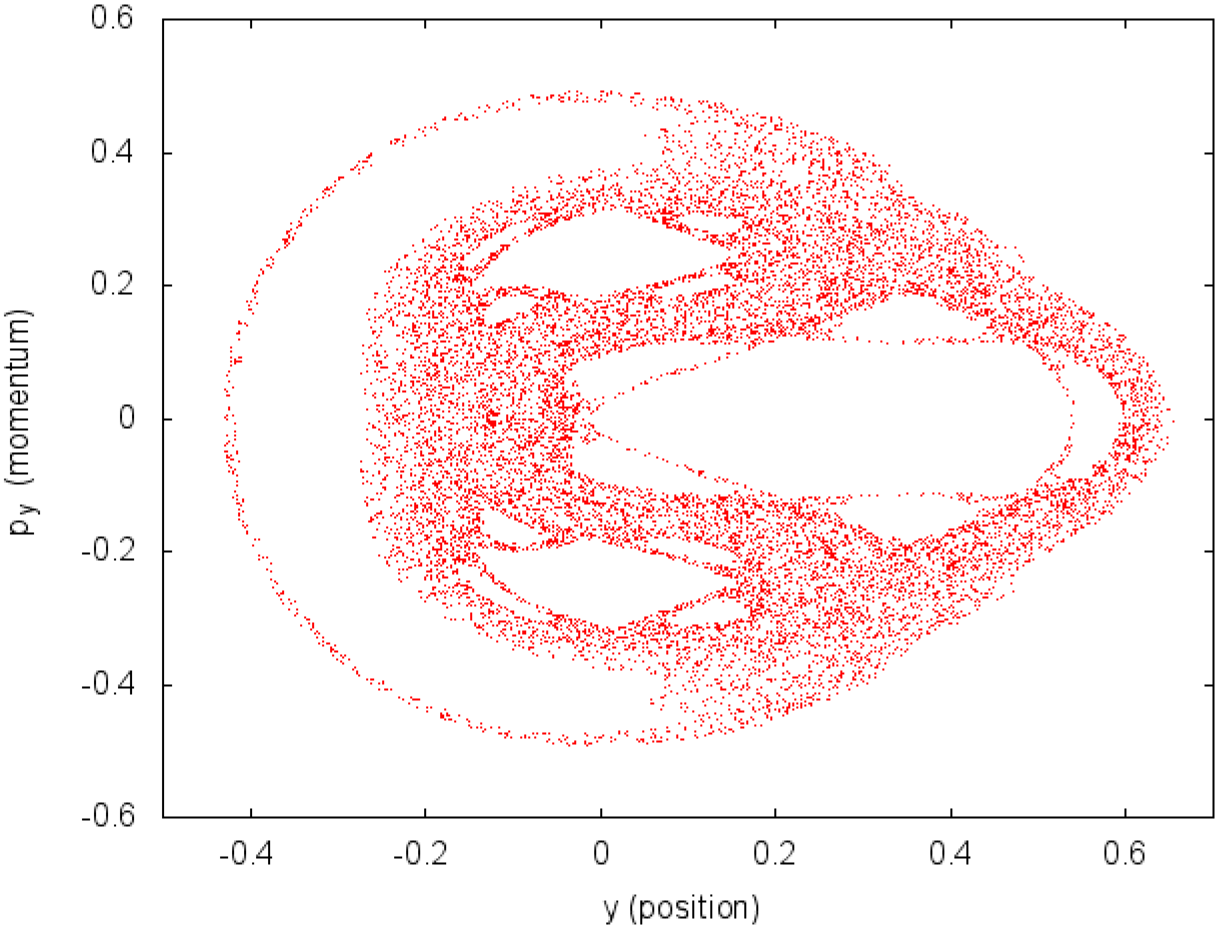
In this sense an N+1 degree of freedom Hamiltonian system corresponds to a 2N-dimensional map.

Hénon-Heiles system: PSS (x=0)



Chaotic orbit

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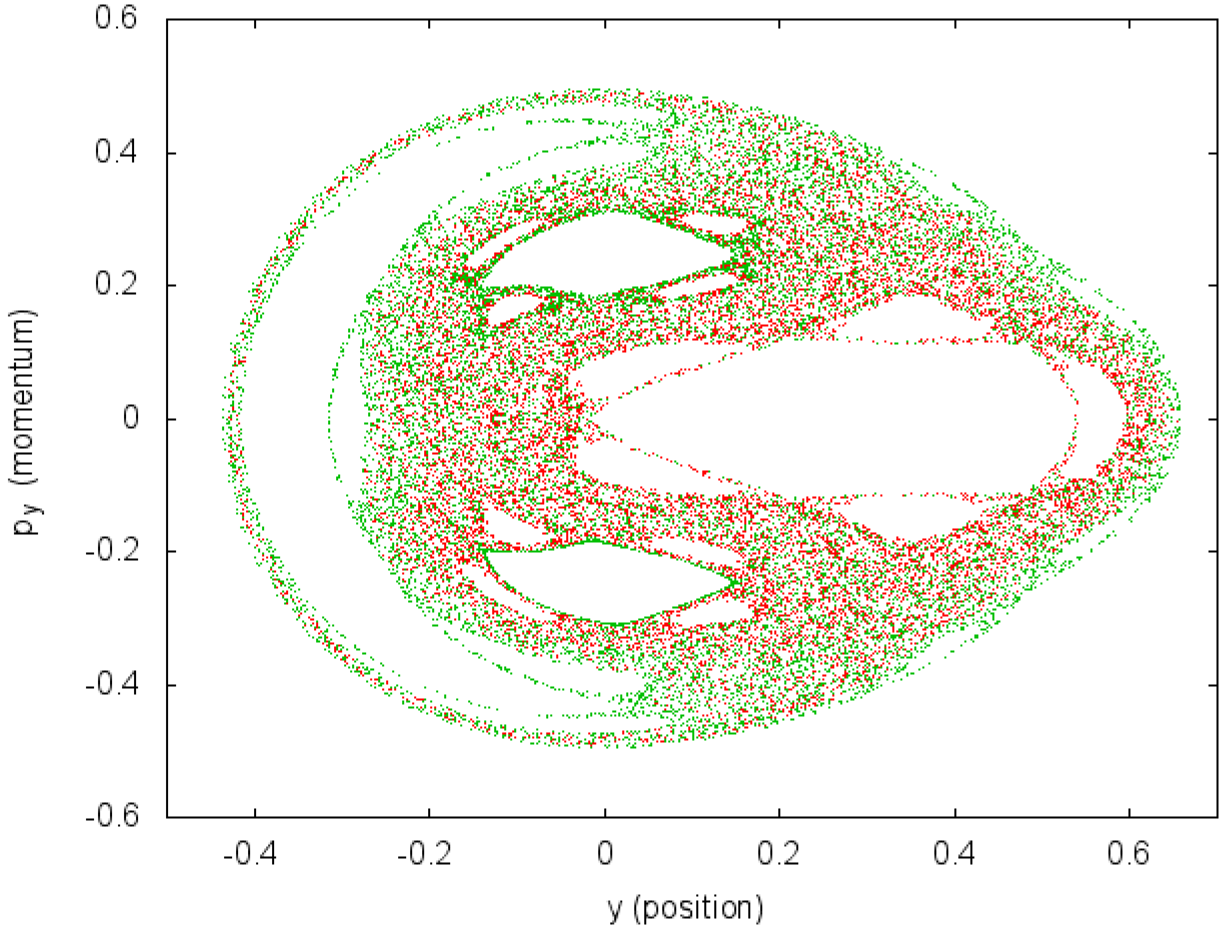

Chaotic orbit - Perturbed chaotic orbit

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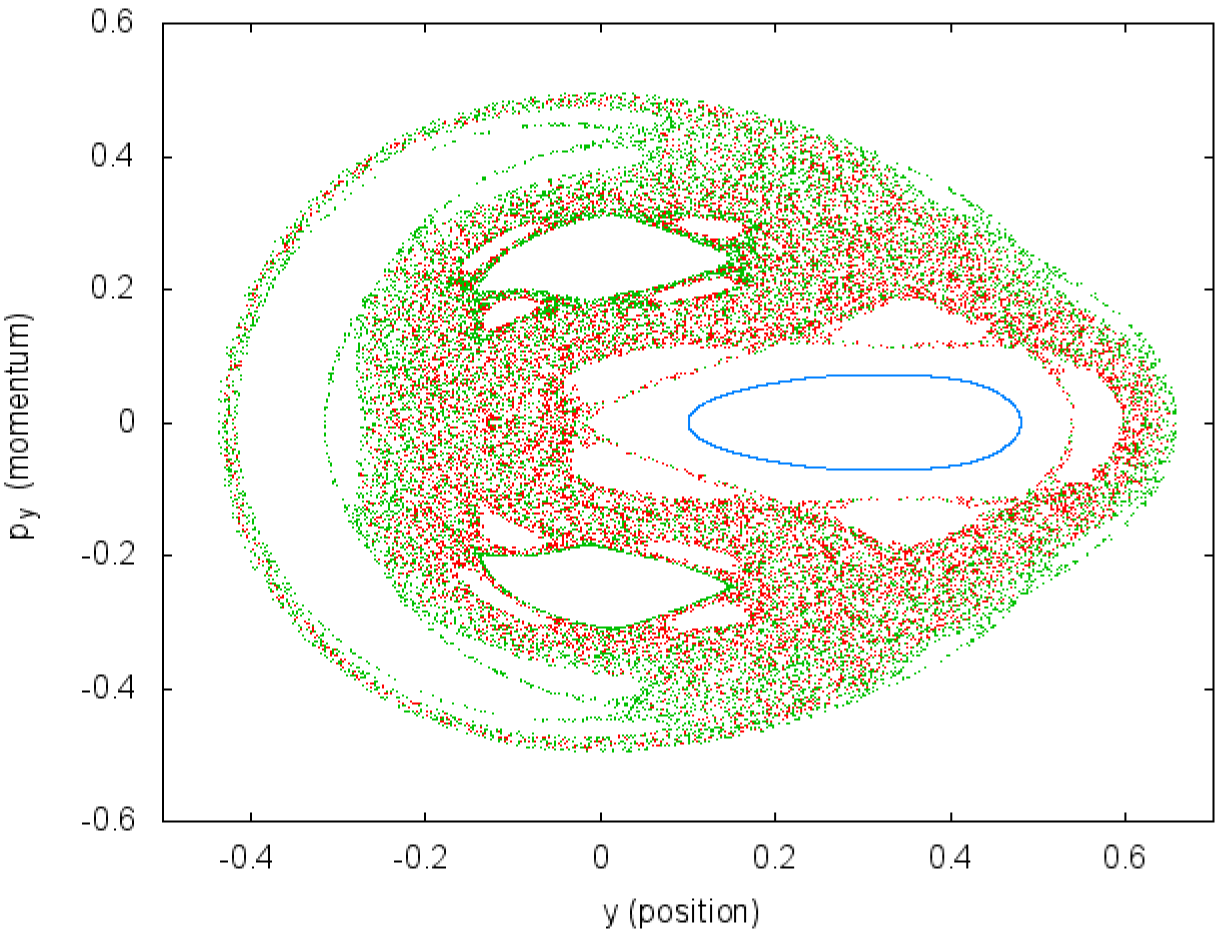

Chaotic orbit - Perturbed chaotic orbit

Regular orbit

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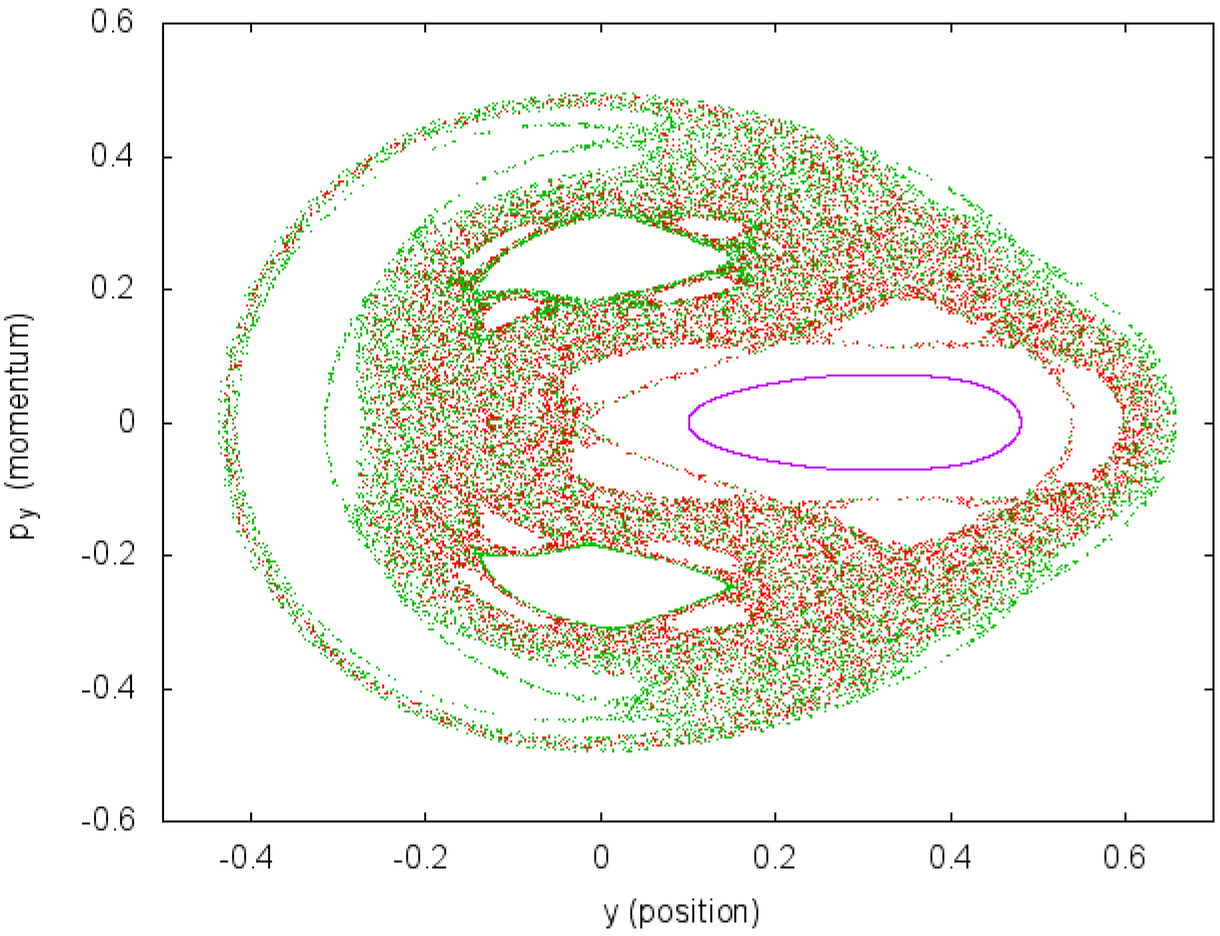

Chaotic orbit - Perturbed chaotic orbit

Regular orbit - Perturbed regular orbit

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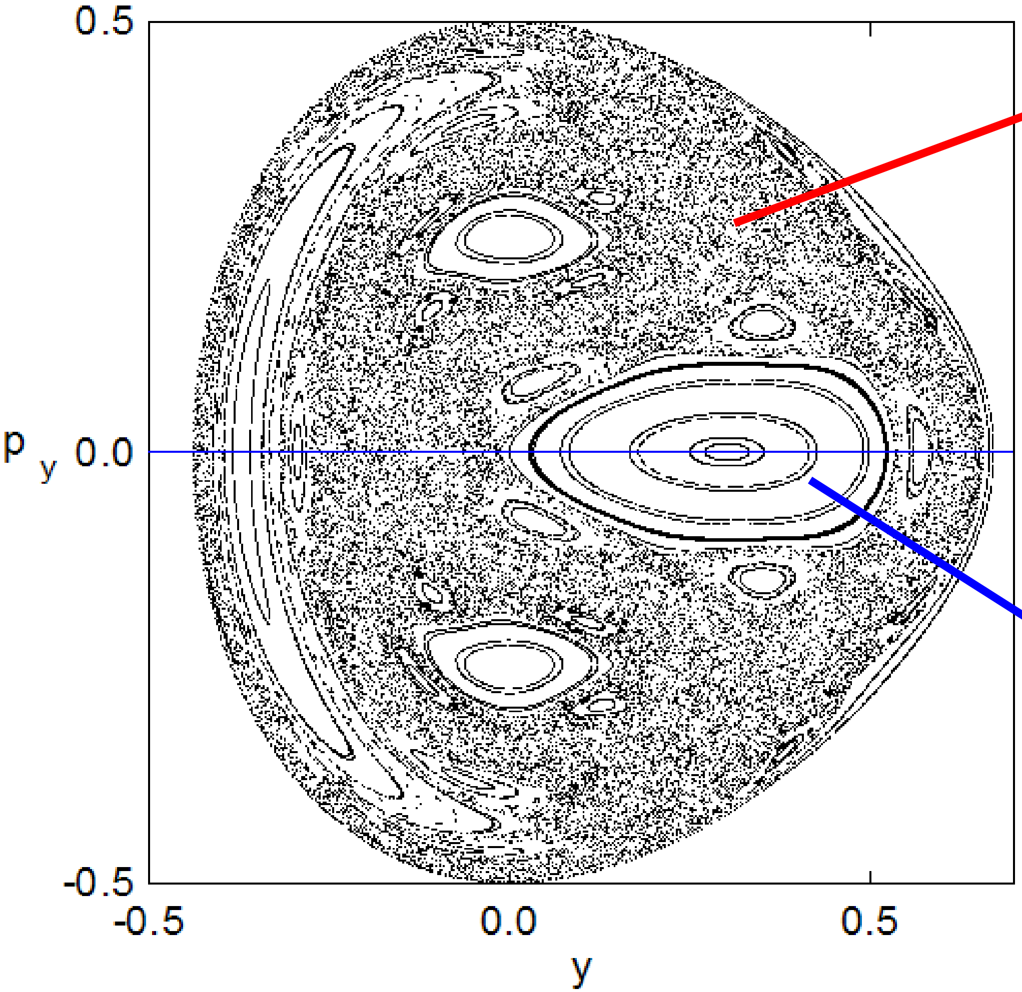


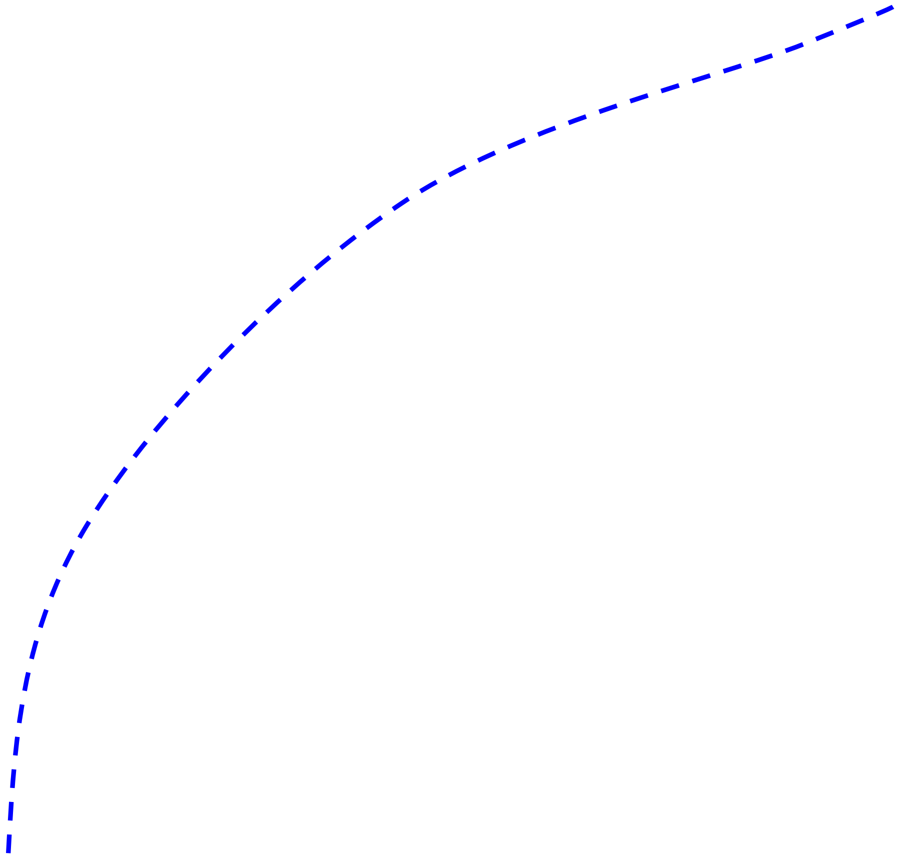
Chaotic motion

Chaotic sea

Regular motion

Island of stability

Computation of the PSS



Computation of the PSS

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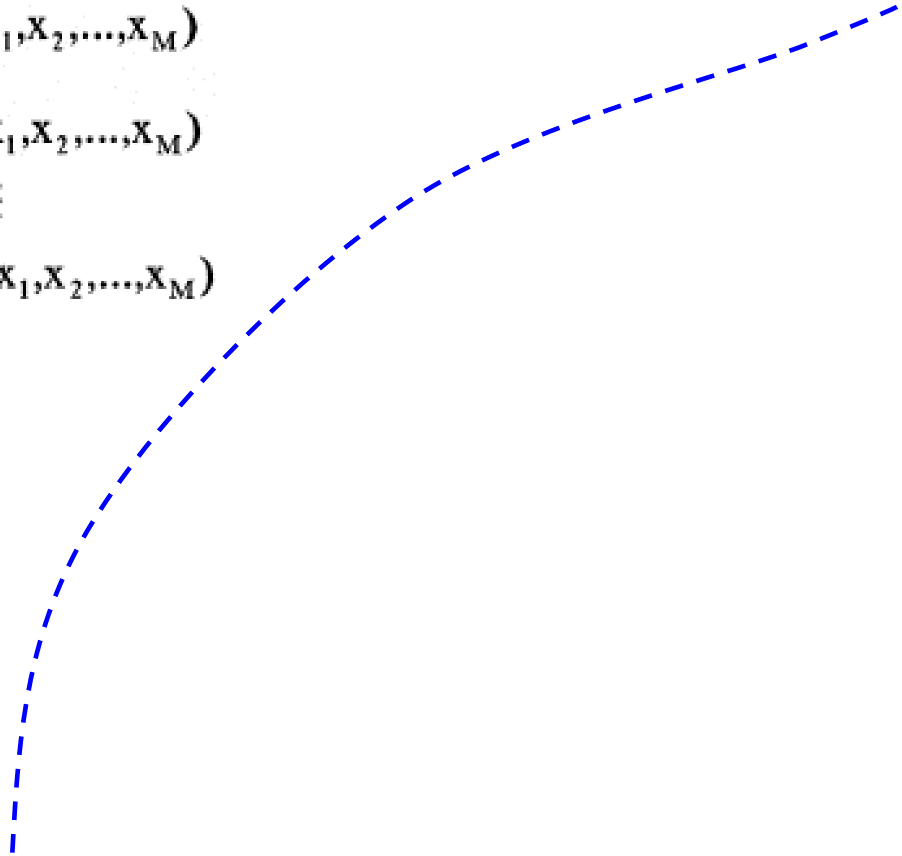

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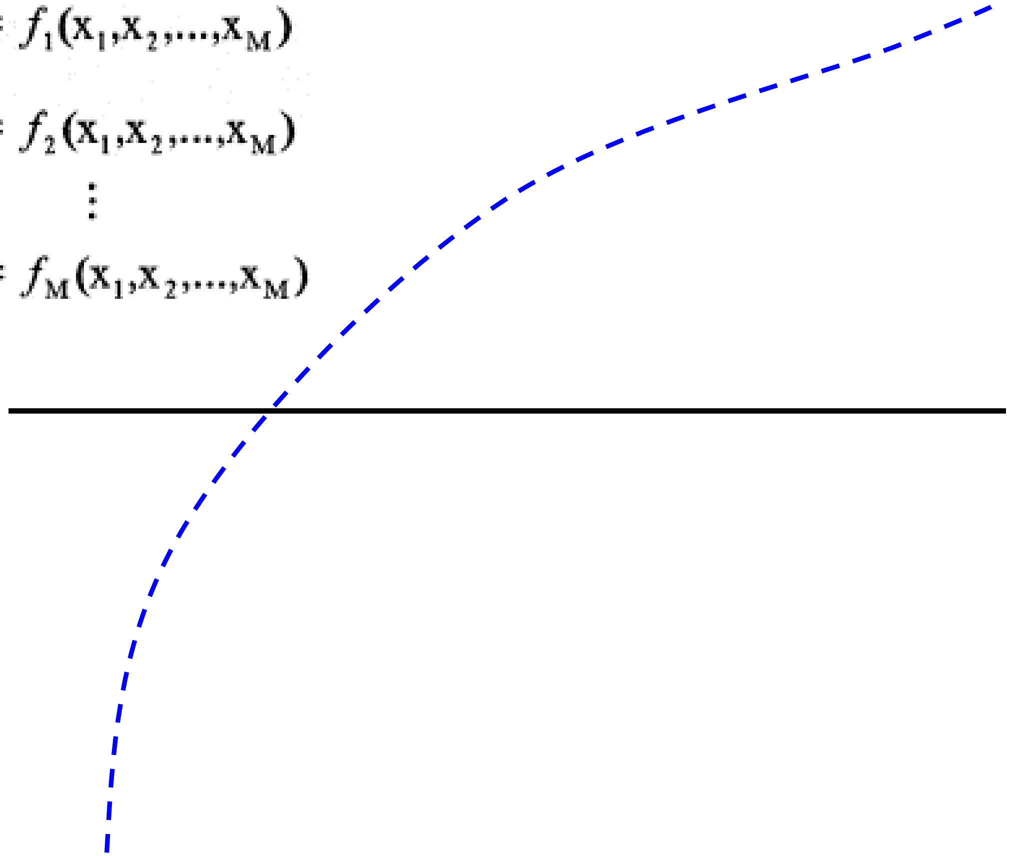

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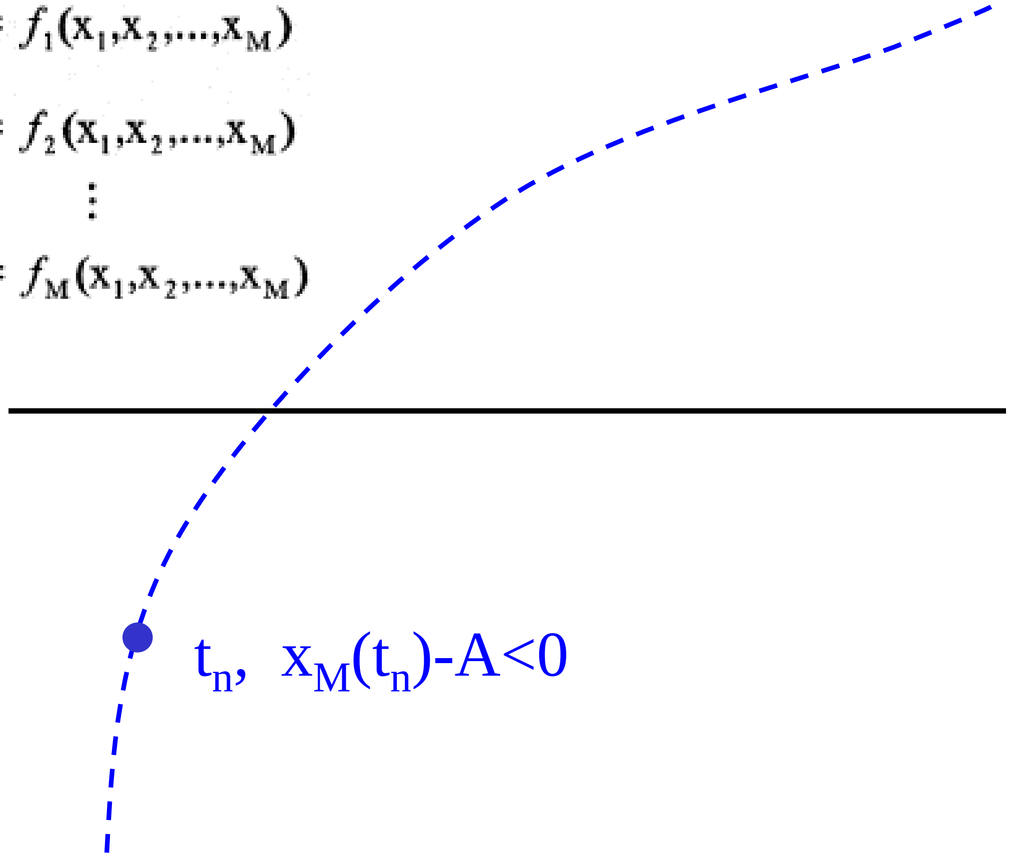

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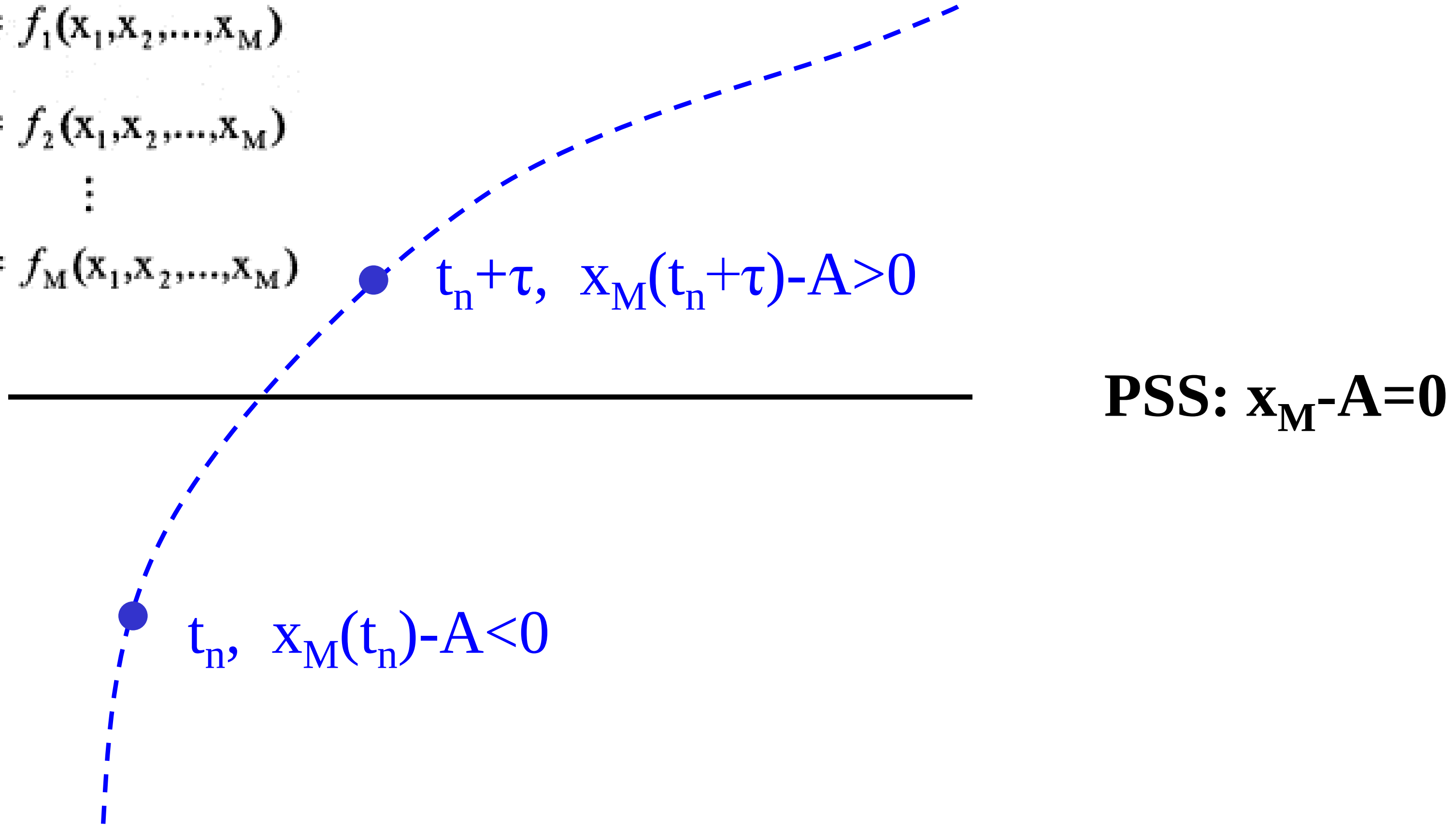

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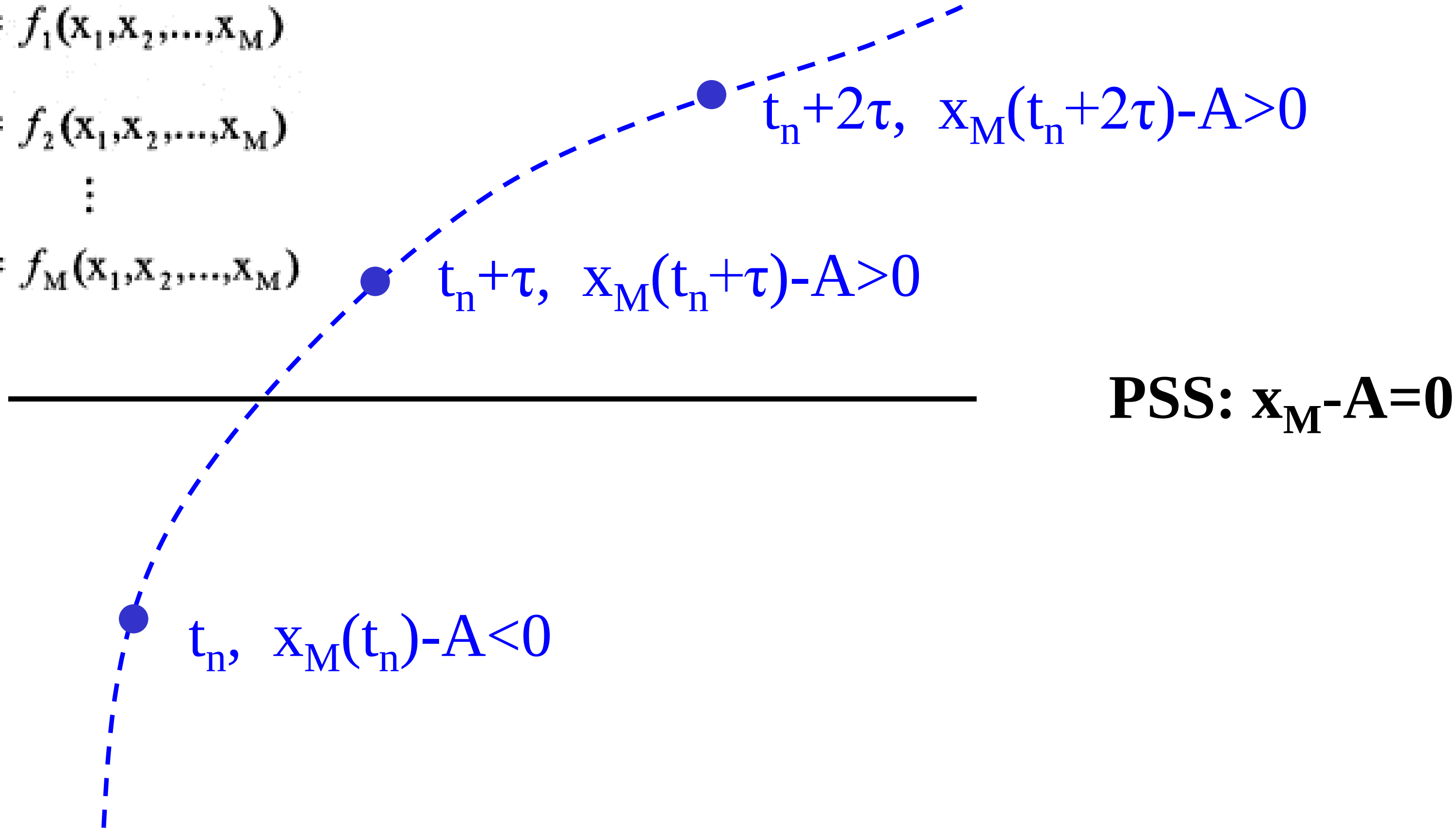

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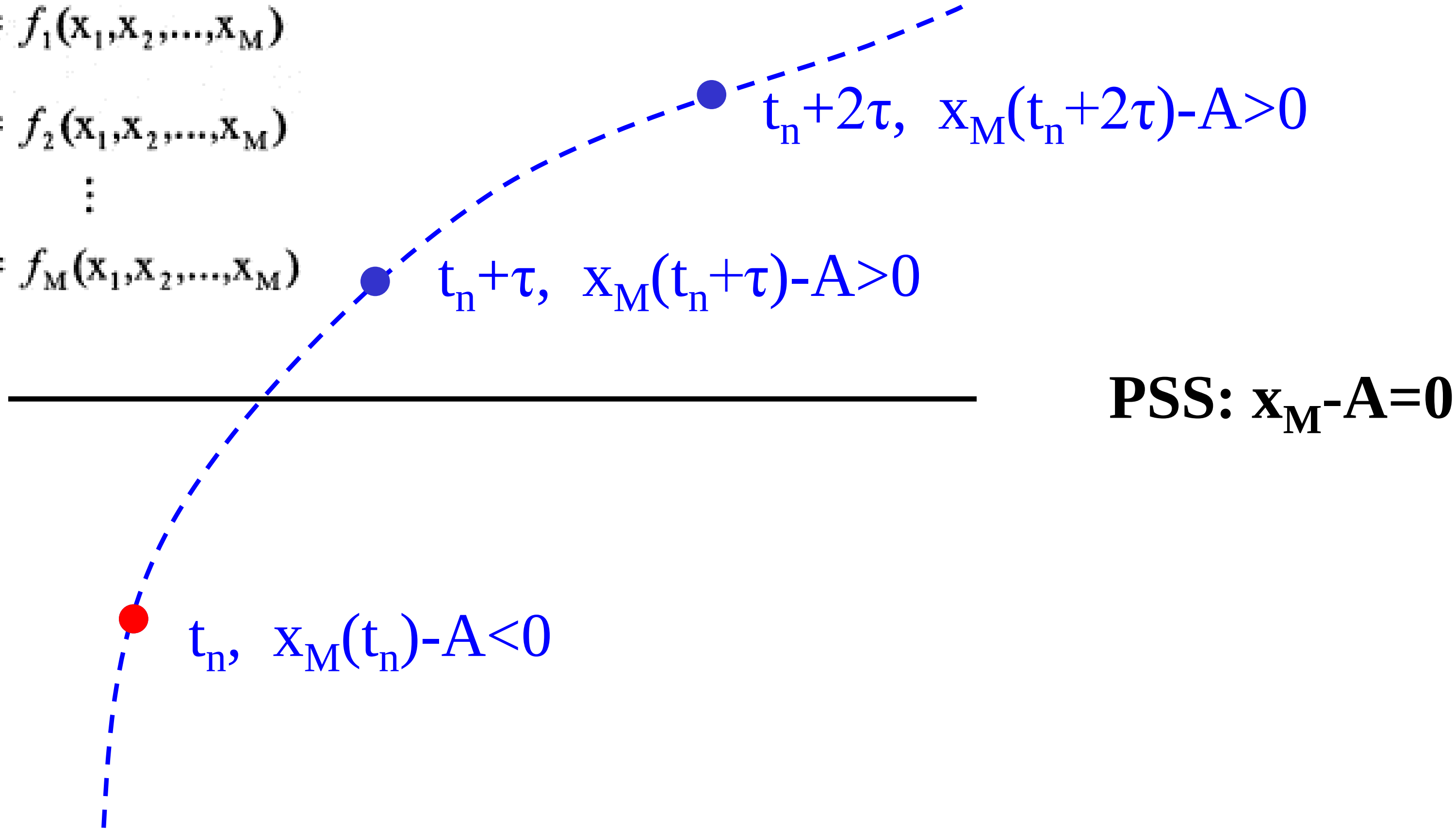

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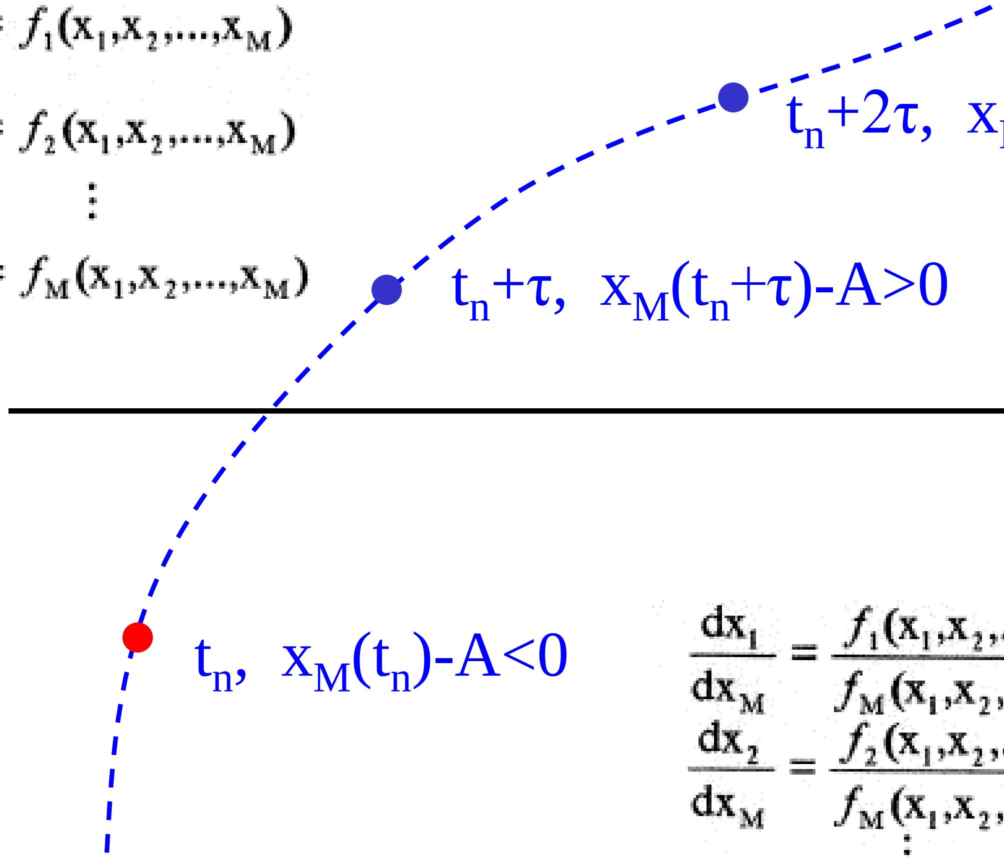

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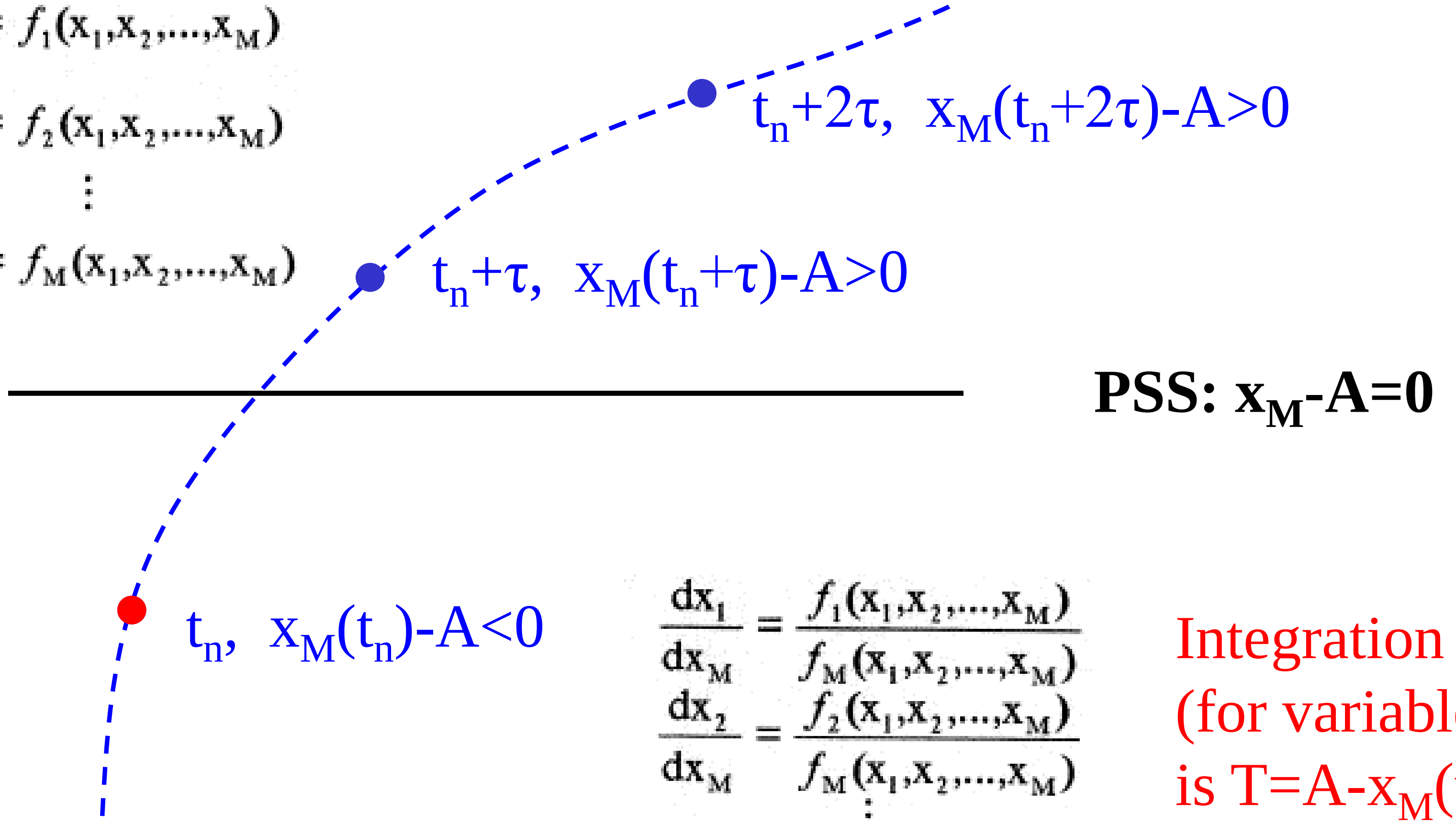

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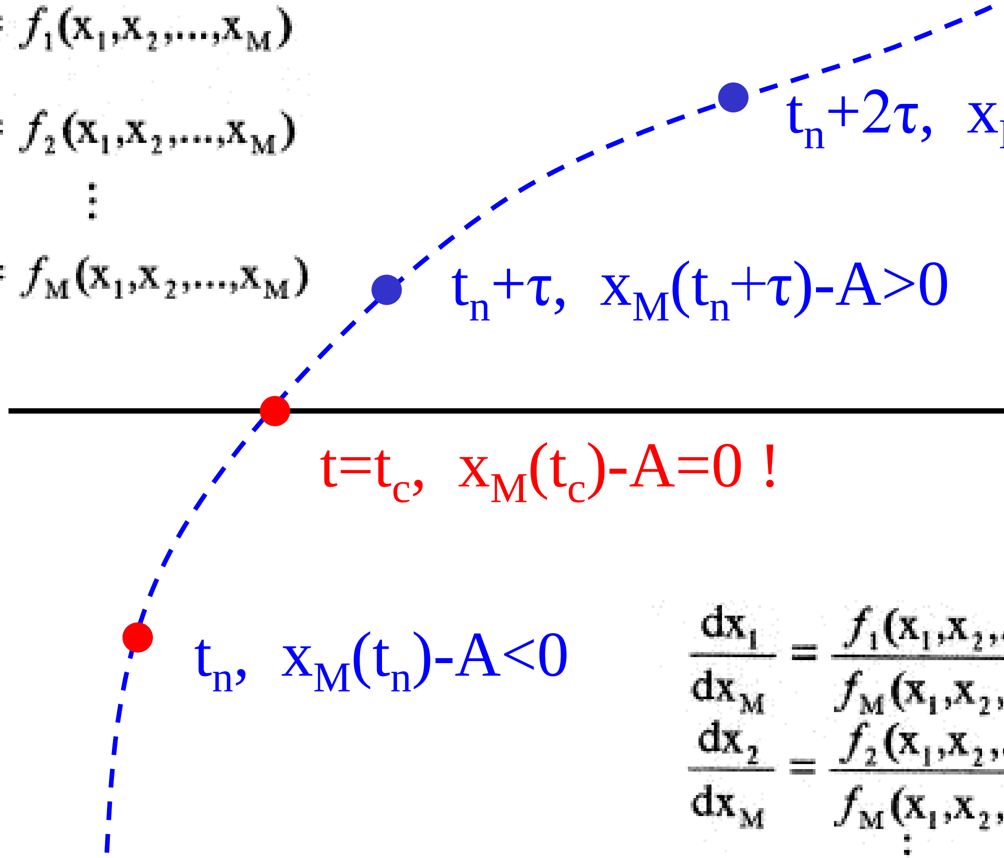

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τ=t,
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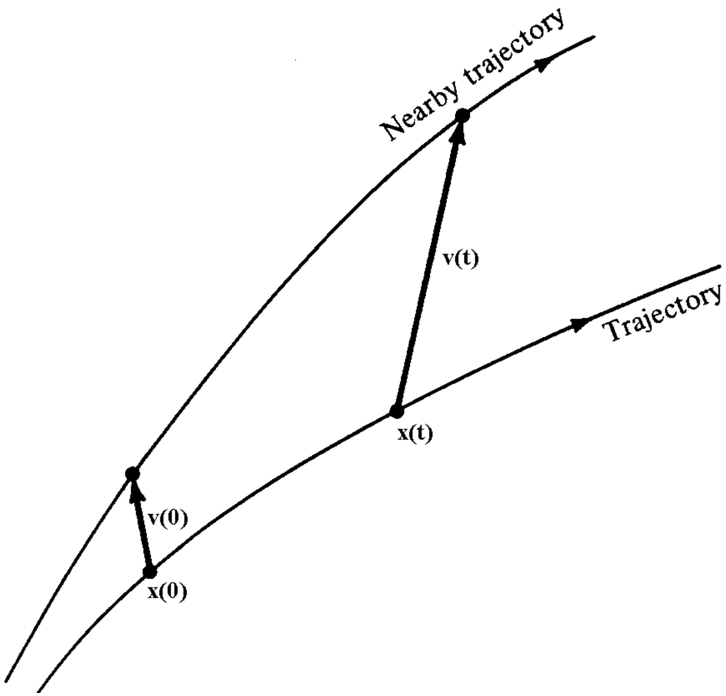
Variational Equations

We use the notation $\mathbf{x} = (q_1,q_2,\ldots,q_N,p_1,p_2,\ldots,p_N)^T$. The **deviation vector** from a given orbit is denoted by

$$\mathbf{v} = (\delta x_1, \delta x_2,\ldots,\delta x_n)^T \text{ , with } n=2N$$



Benettin & Galgani, 1979, in Laval and Gressillon (eds.), op cit, 93

The time evolution of v is given by the so-called **variational equations:**

$$\frac{d\mathbf{v}}{dt} = -\mathbf{J} \cdot \mathbf{P} \cdot \mathbf{v}$$

where

$$\mathbf{J} = \begin{pmatrix} \mathbf{0}_N & -\mathbf{I}_N \\ \mathbf{I}_N & \mathbf{0}_N \end{pmatrix}, \quad \mathbf{P}_{ij} = \frac{\partial^2 H}{\partial x_i \partial x_j} \quad i, j = 1, 2, \ldots, n$$

Example (Hénon-Heiles system)

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + x^2 y - \frac{1}{3}y^3$$

Hamilton's equations of motion:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \Rightarrow \begin{cases} \dot{x} = p_x \\ \dot{y} = p_y \\ \dot{p}_x = -x - 2xy \\ \dot{p}_y = -y - x^2 + y^2 \end{cases}$$

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In order to get the variational equations we linearize the above equations by substituting x, y, px, py with $x+v_1$, $y+v_2$, p_x+v_3, p_y+v_4 where $v=(v_1,v_2,v_3,v_4)$ is the deviation vector. So we get:

$$\dot{p}_x + \dot{v}_3 = -x - v_1 - 2(x + v_1)(y + v_2) \Rightarrow$$

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$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{v}_4 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1+2y & 2x & 0 & 0 \\ 2x & 1-2y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

$$\begin{aligned} \dot{v}_1 &= v_3 \\ \dot{v}_2 &= v_4 \\ \dot{v}_3 &= -v_1 - 2xv_2 - 2yv_1 \\ \dot{v}_4 &= -v_2 - 2xv_1 + 2yv_2 \end{aligned}$$

Example (Hénon-Heiles system)

Variational equations: $$\frac{dv}{dt} = -J \cdot P \cdot v$$

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{v}_4 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1+2y & 2x & 0 & 0 \\ 2x & 1-2y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

$$\begin{aligned} \dot{v}_1 &= v_3 \\ \dot{v}_2 &= v_4 \\ \boxed{\dot{v}_3 = -v_1 - 2xv_2 - 2yv_1} \\ \dot{v}_4 &= -v_2 - 2xv_1 + 2yv_2 \end{aligned}$$

Example (Hénon-Heiles system)

Variational equations: $\dfrac{d\mathbf{v}}{dt} = -\mathbf{J}\cdot\mathbf{P}\cdot\mathbf{v}$

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$$\begin{aligned}\dot v_1 &= v_3\\ \dot v_2 &= v_4\\ \dot v_3 &= -v_1 - 2xv_2 - 2yv_1\\ \dot v_4 &= -v_2 - 2xv_1 + 2yv_2\end{aligned} \quad + \quad \begin{aligned}\dot x &= p_x\\ \dot y &= p_y\\ \dot p_x &= -x - 2xy\\ \dot p_y &= -y - x^2 + y^2\end{aligned}$$

Complete set of equations

Lyapunov Exponents

Roughly speaking, the Lyapunov exponents of a given orbit characterize the mean exponential rate of divergence of trajectories surrounding it.

Consider an orbit in the 2N-dimensional phase space with initial condition x(0) and an initial deviation vector from it v(0). Then the mean exponential rate of divergence is:

$$\sigma(x(0), v(0)) = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|v(t)\|}{\|v(0)\|}$$

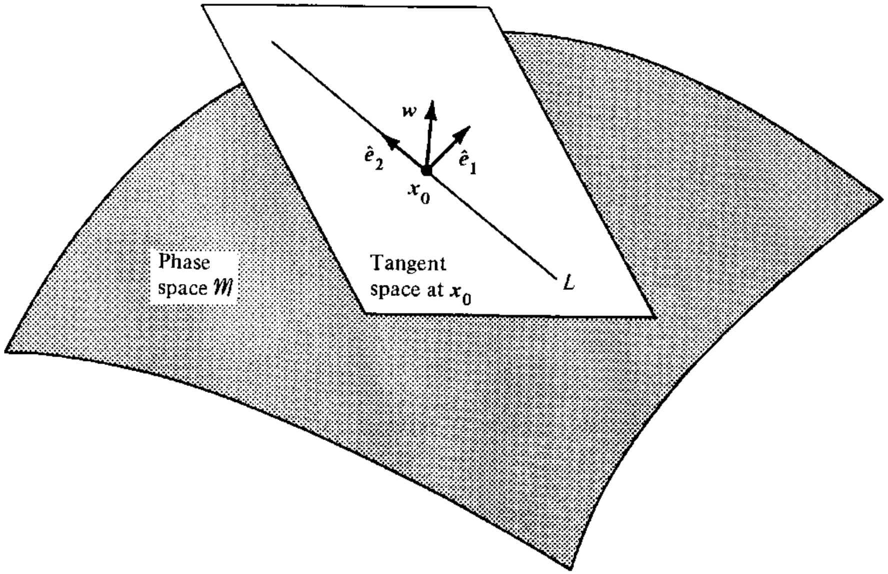
We commonly use the Euclidian norm and set d(0)=||v(0)||=1

Lyapunov Exponents

There exists an M-dimensional basis $\{\hat{e}_i\}$ of v such that for any v, σ takes one of the M (possibly nondistinct) values

$$\sigma_i(x(0)) = \sigma(x(0), \hat{e}_i)$$

which are the Lyapunov exponents.



Benettin & Galgani, 1979, in Laval and Gressillon (eds.), op cit, 93

In autonomous Hamiltonian systems the M exponents are ordered in pairs of opposite sign numbers and two of them are 0.

Computation of the Maximum Lyapunov Exponent

Due to the exponential growth of v(t) (and of d(t)=||v(t)||) we renormalize v(t) from time to time.

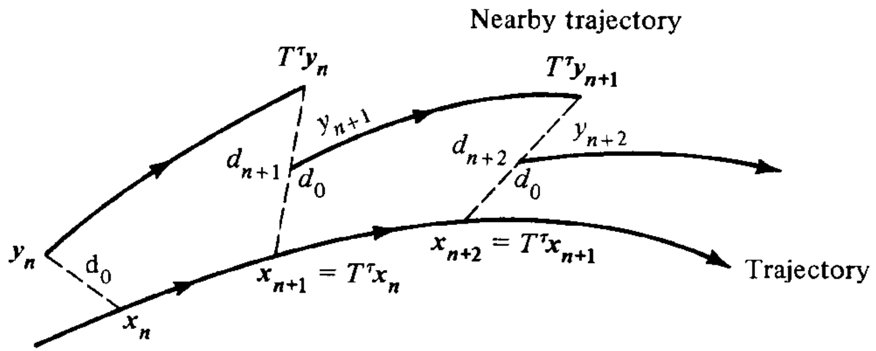


Figure 5.6. Numerical calculation of the maximal Liapunov characteristic exponent. Here $y = x + v$ and τ is a finite interval of time (after Benettin *et al.*, 1976).

Then the Maximum Lyapunov exponent is computed as

$$\sigma_1 = \lim_{n\to\infty} \frac{1}{n\tau} \sum_{i=1}^{n} \ln d_i$$

Maximum Lyapunov Exponent

$\sigma_1 = 0 \rightarrow$ **Regular motion**

$\sigma_1 \neq 0 \rightarrow$ **Chaotic motion**

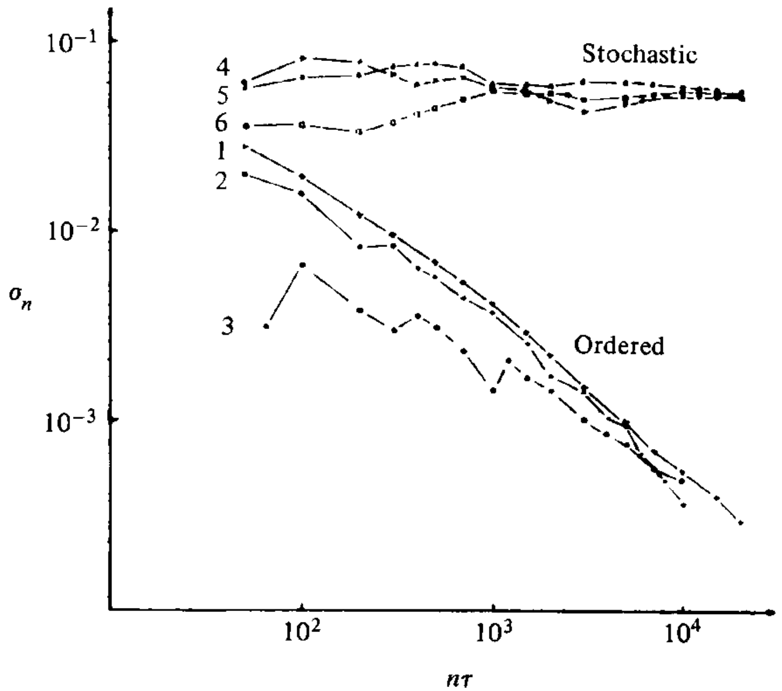
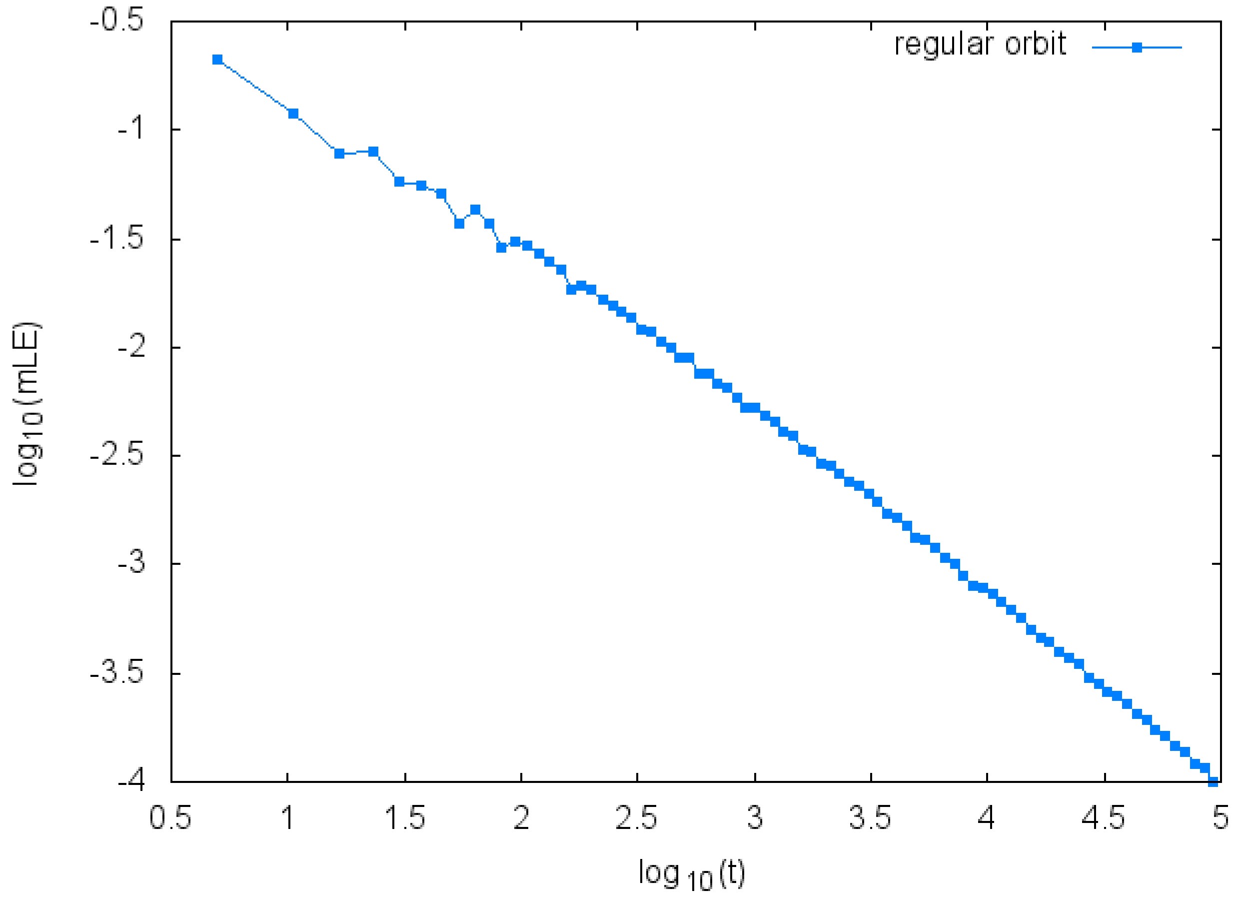


Figure 5.7. Behavior of σ_n at the intermediate energy $E = 0.125$ for initial points taken in the ordered (curves 1–3) or stochastic (curves 4–6) regions (after Benettin *et al.*, 1976).

If we start with more than one linearly independent deviation vectors they will align to the direction defined by the largest Lyapunov exponent for chaotic orbits.

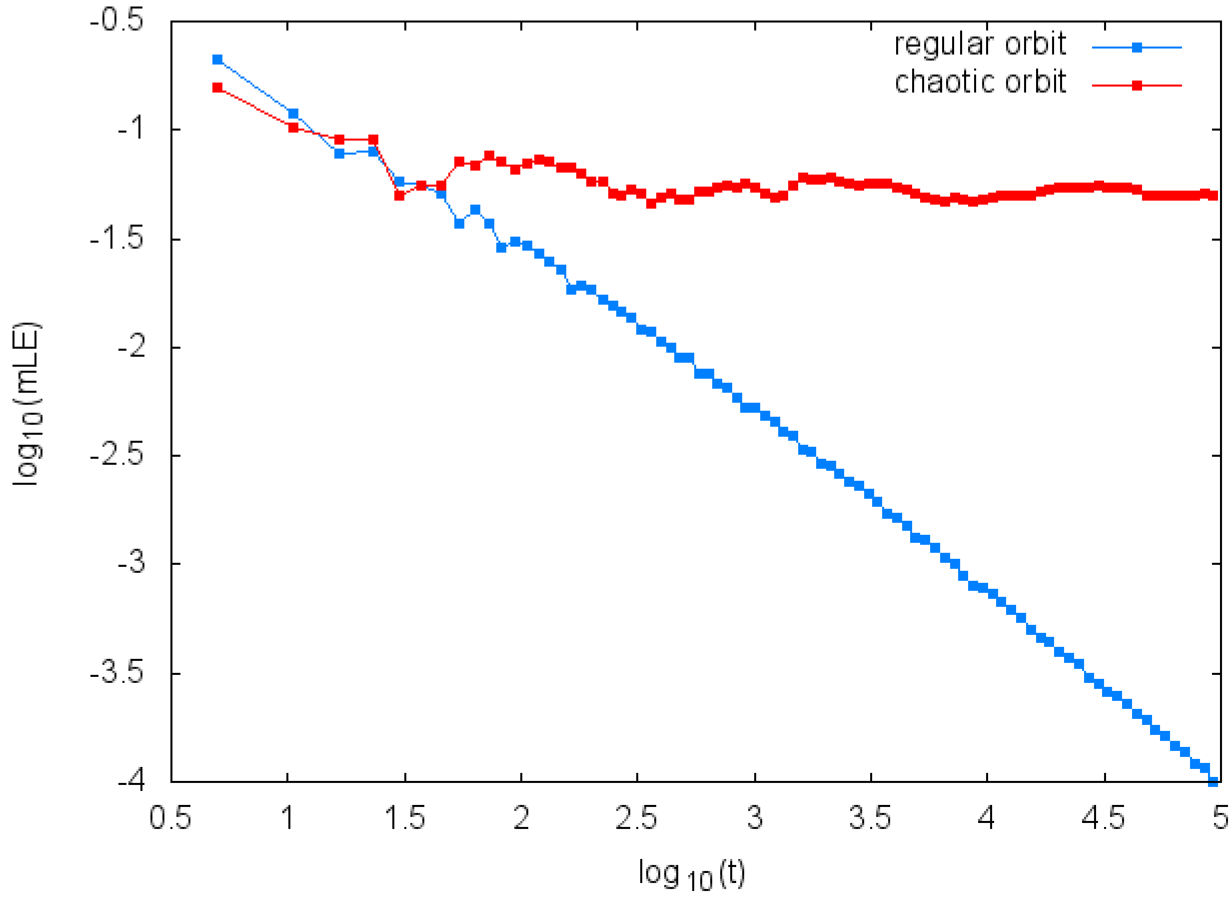
Maximum Lyapunov Exponent

Hénon-Heiles system: Regular orbit



Maximum Lyapunov Exponent

Hénon-Heiles system: Regular orbit and Chaotic orbit



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